

# KLEIN'S PROGRAMME AND QUANTUM MECHANICS

JESÚS CLEMENTE-GALLARDO AND GIUSEPPE MARMO

**ABSTRACT.** We review the geometrical formulation of Quantum Mechanics to identify, according to Klein's programme, the corresponding group of transformations. For closed systems, it is the unitary group. For open quantum systems, the semigroup of Kraus maps contains, as a maximal subgroup, the general linear group. The same group emerges as the exponentiation of the  $C^*$ -algebra associated with the quantum system, when thought of as a Lie algebra. Thus, open quantum systems seem to identify the general linear group as associated with quantum mechanics and moreover suggest to extend the Klein programme also to groupoids. The usual unitary group emerges as a maximal compact subgroup of the general linear group.

Geometric Quantum Mechanics; Open quantum systems; Klein Programme

## 1. INTRODUCTION: GEOMETRIC DESCRIPTION OF QUANTUM MECHANICS

The Erlangen Programme (see [12]) was above all an affirmation of the key role of groups in geometry. Klein writes: "In the following third part of my collected works ... are those papers involving the concept of continuous group of transformations". He announced the main aspects of his programme as follows: "Given a manifold and a transformation group acting on it, to investigate those properties of figures on that manifold which are invariant under all transformations of that group."

In modern language, we could say that each geometry or geometrical structure is fully characterized by a subgroup of the diffeomorphism group. Different realizations of the same group give rise to isomorphic geometrical structures or geometries, i.e., a "geometry" is associated with an abstract (Lie) group and not with a specific realization.

Therefore, a geometrical description of Quantum Mechanics should identify an associated group. In this paper, to be able to deal with finite dimensional Lie groups and corresponding geometrical structures we shall restrict our considerations to systems with finite dimensional manifolds of observable and states. Our presentation should be thought of as providing guidelines to deal with the more realistic situation of infinite dimensional states, which would require a theory of infinite dimensional Lie groups, not fully available at present time.

The main point of our paper is that if one takes the  $C^*$ -algebraic approach to quantum mechanics, the general complex linear group of transformations emerges as a more fundamental group than its maximal compact subgroup of unitary transformations.

To contextualize the search for the group to be associated with Quantum Mechanics, we shall consider the two main pictures usually encountered in the description of quantum systems, Schrödinger's and Heisenberg's, from a geometric perspective. We shall follow the construction which has been developed in the last years (see for example [2, 3, 4, 5, 6] and references therein). As a running example we shall consider the case of a two-level system (or a qubit), and we shall discuss most of the emerging geometrical structures in this particular situation.

## 2. GEOMETRIC QUANTUM MECHANICS

**2.1. The Schrödinger picture.** In the Schrödinger picture, the carrier space  $\mathcal{R}(\mathcal{H})$  which represents the states of the model is the projective space corresponding to the Hilbert space  $\mathcal{H}$  and, as such, a Kähler manifold. Therefore the geometrical information of this picture is contained in a Riemannian structure, a symplectic structure and a complex one, which are compatible. Each two of them determines the third one. This is in agreement with the observation that for linear invertible transformations,

$$\begin{aligned} \text{symplectic} \cap \text{Riemannian} &\simeq \text{unitary} \\ \text{complex} \cap \text{Riemannian} &\simeq \text{unitary} \\ \text{symplectic} \cap \text{complex} &\simeq \text{unitary} \end{aligned}$$

**2.1.1. The Hilbert space formulation.** The various algebraic structures take the form of tensors defined on the carrier space. Thus we take the complex vector space of a  $n$ -level quantum system  $\mathcal{H} = \mathbb{C}^n$  and consider it as a Kähler manifold: a real vector space  $\mathbb{R}^{2n}$  with a complex structure  $J$  (a  $(1, 1)$ -tensor field), a Riemannian structure  $g$  and a symplectic form  $\omega$ , related as

$$(2.1) \quad g(X, Y) = \omega(X, JY), \quad \forall X, Y \in \mathfrak{X}(M).$$

Analogously we can consider the corresponding contravariant tensors  $G$  and  $\Lambda$ .

If we consider a two level system, the Hilbert space corresponds to  $\mathcal{H} = \mathbb{C}^2$  with the usual Hermitian structure

$$(2.2) \quad \langle z|w \rangle = \sum_k \bar{z}^k w^k.$$

To consider it as a real vector space we can consider a basis and associated coordinates  $\{z^1, z^2\}$ . The real carrier space becomes  $\mathbb{R}^4$  with coordinates  $(q^1, p_1, q^2, p_2)$ , which are the real ( $q^k$ ) and imaginary ( $p_k$ ) parts of the complex coordinate  $z^k$ , for  $k = 1, 2$ . In these coordinates, the expression of the tensors defining the Kähler structure would be as follows:

- the Riemannian metric corresponds to an Euclidean metric on  $\mathbb{R}^4$ :

$$(2.3) \quad g = dq^1 \otimes dq^1 + dp_1 \otimes dp_1 + dq^2 \otimes dq^2 + dp_2 \otimes dp_2$$

or as a bidifferential operator, contravariant  $(0, 2)$  tensor field:

$$(2.4) \quad G = \frac{\partial}{\partial q^1} \otimes \frac{\partial}{\partial q^1} + \frac{\partial}{\partial p_1} \otimes \frac{\partial}{\partial p_1} + \frac{\partial}{\partial q^2} \otimes \frac{\partial}{\partial q^2} + \frac{\partial}{\partial p_2} \otimes \frac{\partial}{\partial p_2}.$$

- analogously, the symplectic structure can be written as a 2-form

$$(2.5) \quad \omega = dq^1 \wedge dp_1 + dq^2 \wedge dp_2$$

or as a Poisson bivector field

$$(2.6) \quad \Omega = \frac{\partial}{\partial q^1} \wedge \frac{\partial}{\partial p_1} + \frac{\partial}{\partial q^2} \wedge \frac{\partial}{\partial p_2}.$$

- finally, the complex structure is written as a  $(1-1)$  tensor field

$$(2.7) \quad J = dq^1 \otimes \frac{\partial}{\partial p_1} - dp_1 \otimes \frac{\partial}{\partial q^1} + dq^2 \otimes \frac{\partial}{\partial p_2} - dp_2 \otimes \frac{\partial}{\partial q^2}$$

As for the operators on  $\mathcal{H}$  we can also translate them into tensorial terms in several ways. One of them is to define the quadratic functions corresponding to them and write, for any  $A \in \mathfrak{gl}(\mathcal{H})$  the associated function

$$(2.8) \quad f_A(\psi) = \frac{1}{2} \langle \psi | A \psi \rangle.$$

It is immediate to see that the quadratic functions corresponding to Hermitian operators are real valued, in the general case they are complex. We shall denote by  $\mathcal{F}_2(\mathcal{H})$  and  $\mathcal{F}_2^{\mathbb{R}}(\mathcal{H})$  the set of quadratic complex valued functions and the subset of quadratic real valued functions respectively.

For instance, in our example  $\mathcal{H} = \mathbb{C}^2 \simeq \mathbb{R}^4$  we can consider the set of  $2 \times 2$  complex matrices which is a  $C^*$ -algebra,  $\mathfrak{gl}(2, \mathbb{C})$  and contains  $\mathrm{GL}(2, \mathbb{C})$  and the real elements correspond to Hermitian matrices, which, after multiplication by the imaginary unit, would correspond to elements in  $\mathfrak{u}(2)$ . The quadratic functions corresponding to the Pauli matrices, for instance, would be written as

$$(2.9) \quad f_{\sigma_0}(\psi) = \frac{1}{2} \left( (q^1)^2 + (q^2)^2 + p_1^2 + p_2^2 \right),$$

$$(2.10) \quad f_{\sigma_1}(\psi(q, p)) = \frac{1}{2} \langle \psi | \sigma_1 \psi \rangle = (q^1 q^2 + p^1 p^2),$$

$$(2.11) \quad f_{\sigma_2}(\psi(q, p)) = \frac{1}{2} \langle \psi | \sigma_2 \psi \rangle = (q^1 p_2 - q^2 p_1),$$

and

$$(2.12) \quad f_{\sigma_3}(\psi) = \frac{1}{2} \langle \psi | \sigma_3 \psi \rangle = \frac{1}{2} \left( (q^1)^2 - (q^2)^2 + p_1^2 - p_2^2 \right)$$

where  $\sigma_0 = \mathbb{I}_2$ , the identity matrix in two dimensions.

On  $\mathcal{F}_2(\mathcal{H})$  we can export the algebraic structures the set  $\mathfrak{gl}(\mathcal{H})$  is endowed with:

- the natural Lie algebra structure associated with the commutator  $[A, B]_{\mathfrak{gl}} = (AB - BA)$  is not the most convenient since it does not define a subalgebra structure on the set of Hermitian operators. Instead, we can consider  $[A, B] = -i(AB - BA)$  which does define a Lie algebra structure on  $\mathrm{Herm}(\mathcal{H})$  and that can be realized on  $\mathcal{F}_2(\mathcal{H})$  by using the Poisson tensor (2.6):

$$(2.13) \quad f_{[A, B]} = \Omega(df_A, df_B) := \{f_A, f_B\}; \quad \forall A, B \in \mathrm{Herm}(\mathcal{H}).$$

Last equation is defining a Poisson bracket  $\{\cdot, \cdot\}$  on  $\mathcal{F}_2^{\mathbb{R}}$ .

- The Jordan structure associated to the anti-commutator  $[A, B]_+ = AB + BA$  is realized by using the symmetric tensor (2.4):

$$(2.14) \quad f_{[A, B]_+} = G(df_A, df_B) := \{f_A, f_B\}_+; \quad \forall A, B \in \mathfrak{gl}(\mathcal{H}).$$

Last equation is defining a Jordan bracket  $\{\cdot, \cdot\}_+$  on  $\mathcal{F}_2(\mathcal{H})$ . This operation is inner on  $\mathcal{F}_2^{\mathbb{R}}$ .

- The associative product of  $\mathfrak{gl}(\mathcal{H})$  can be written by using a combination of both operations above. Thus as  $AB = \frac{1}{2}[A, B]_+ + \frac{i}{2}[A, B]$ , we can introduce then a  $\star$ -product on the quadratic functions

$$(2.15) \quad f_{AB} = \frac{1}{2}G(df_A, df_B) + \frac{i}{2}\Omega(df_A, df_B) := f_A \star f_B; \quad \forall A, B \in \mathrm{Herm}(\mathcal{H}).$$

If we extend it by linearity, last equation is defining a new operation  $\star$  on  $\mathcal{F}_2(\mathcal{H})$ , which is non-local and non-commutative, but it is associative.

With respect to these tensors, we can define two types of vector fields associated with quadratic functions (and hence to the elements of  $\mathfrak{gl}(\mathcal{H})$ ):

- Hamiltonian vector fields, associated with quadratic functions via the Poisson structure:

$$(2.16) \quad X_{f_A} = \Omega(df_A, \cdot) = \{f_A, \cdot\}, \quad f_A \in \mathcal{F}_2(\mathcal{H}).$$

- And gradient vector fields associated with quadratic functions via the symmetric tensor  $G$ :

$$(2.17) \quad Y_{f_A} = G(df_A, \cdot) = \{f_A, \cdot\}_+, \quad f_A \in \mathcal{F}_2(\mathcal{H}).$$

Gradient vector fields associated with the Pauli matrices via the quadratic functions on  $\mathcal{H} = \mathbb{C}^2 \simeq \mathbb{R}^4$  are given by:

$$(2.18) \quad Y_{f_1} = q^2 \frac{\partial}{\partial q^1} + q^1 \frac{\partial}{\partial q^2} + p_2 \frac{\partial}{\partial p_1} + p_1 \frac{\partial}{\partial p_2}$$

$$(2.19) \quad Y_{f_2} = p_2 \frac{\partial}{\partial q^1} + q^1 \frac{\partial}{\partial p_2} - p_1 \frac{\partial}{\partial q^2} + q^2 \frac{\partial}{\partial p_1}$$

$$(2.20) \quad Y_{f_3} = q^1 \frac{\partial}{\partial q^1} - q^2 \frac{\partial}{\partial q^2} + p_1 \frac{\partial}{\partial p_1} - p_2 \frac{\partial}{\partial p_2}$$

while for the corresponding Hamiltonian vector fields we find

$$(2.21) \quad X_{f_1} = q^2 \frac{\partial}{\partial p_1} + q^1 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial q^1} - p_1 \frac{\partial}{\partial q^2}$$

$$(2.22) \quad X_{f_2} = p_2 \frac{\partial}{\partial p_1} - q^1 \frac{\partial}{\partial q^2} - p_1 \frac{\partial}{\partial p_2} - q^2 \frac{\partial}{\partial q^1}$$

$$(2.23) \quad X_{f_3} = q^1 \frac{\partial}{\partial p_1} - q^2 \frac{\partial}{\partial p_2} - p_1 \frac{\partial}{\partial q^1} + p_2 \frac{\partial}{\partial q^2}$$

From the compatibility conditions (Eq. (2.1)) of the tensors of the Kähler structure, it is clear that

$$(2.24) \quad Y_{f_A} = -J(X_{f_A}).$$

We notice that the Hamiltonian vector fields close on the Lie algebra of  $\mathfrak{su}(2)$  while the three gradient vector fields transform like a vector under the action of the Hamiltonian ones. The commutator of two gradient vector fields is not of gradient type, but it turns out to be a Hamiltonian vector field.

**2.1.2. The projective space formulation.** As it is well known the probabilistic interpretation of quantum Mechanics requires that (pure) states are described by rays and not by vectors. Therefore the set of pure states is described by the complex projective space  $\mathcal{PH}$ . To transfer the constructed tensorial description to the projective space we must first write the relevant objects in a geometric language. To this aim, we consider two vector fields

- the dilation vector field on  $\mathcal{H}$ , denoted as  $\Delta$ , which in coordinates  $(q^j, p_k)$  reads

$$(2.25) \quad \Delta = \sum_k \left( q^k \frac{\partial}{\partial q^k} + p_k \frac{\partial}{\partial p_k} \right),$$

- and the vector field  $\Gamma = J(\Delta)$ , representing the global phase change on the Hilbert space  $\mathcal{H}$  and which in local coordinates reads

$$(2.26) \quad \Gamma = \sum_k \left( q^k \frac{\partial}{\partial p_k} - p_k \frac{\partial}{\partial q^k} \right)$$

As they define an integrable distribution we can consider the corresponding foliation  $\mathcal{P}$ , the manifold of leaves represents  $\mathcal{PH}$  in the geometric language. The corresponding fibration  $\pi : \mathcal{H}_0 \rightarrow \mathcal{P}$  on the projective space identifies projectable tensors on  $\mathcal{H}_0 = \mathcal{H} - \{0\}$  as those which are defined on the space of pure states. With this identification we can avoid using the different charts required to work on  $\mathcal{P}$  (as it is no longer a vector space), and use objects on  $\mathcal{H}$ , as long as they are projectable with respect to  $\pi$ .

Both vector fields have interesting properties. Just to mention some of them, we can consider their relations with respect to the Kähler structure:

- The norm of the vector fields  $\Gamma$  and  $\Delta$  at any point equals the norm of the (vector) point at which they are evaluated, i.e.

$$(2.27) \quad g(\psi)(\Delta(\psi), \Delta(\psi)) = \langle \psi | \psi \rangle = g(\psi)(\Gamma(\psi), \Gamma(\psi)).$$

- Analogously, the symplectic area generated by them at a point  $\psi \in \mathcal{H}$  is equal to the norm of the vector (point)

$$(2.28) \quad \omega(\Delta, \Gamma) = \omega(\Delta, J(\Delta)) = g(\Delta, \Delta) = \langle \psi | \psi \rangle.$$

- The two vector fields are orthogonal with respect to  $g$ :

$$(2.29) \quad g(\Delta, \Gamma) = g(\Delta, J(\Delta)) = \omega(\Delta, J^2(\Delta)) = -\omega(\Delta, \Delta) = 0,$$

where we used the compatibility condition of the Kähler structure (Eq. (2.1)). This is a general property for any two vector fields of the form  $X$  and  $J(X)$ , they are always orthogonal to each other. We shall come back to this point later.

- They are the gradient and Hamiltonian vector fields associated to the quadratic function corresponding the identity matrix, i.e., given  $f_0 = \frac{1}{2} \sum_k (q^k)^2 + p_k^2$ , we have

$$(2.30) \quad Y_{f_0} = \Delta = \sum_k \left( q^k \frac{\partial}{\partial q^k} + p_k \frac{\partial}{\partial p_k} \right),$$

and

$$(2.31) \quad X_{f_0} = \Gamma = J(\Delta) = \sum_k \left( q^k \frac{\partial}{\partial p_k} - p_k \frac{\partial}{\partial q^k} \right).$$

In the example of the two level system, the projective space  $\mathbb{CP}^1$  is diffeomorphic to the sphere  $S^2$ , and the corresponding fibration  $\mathbb{C}_0^2 \simeq \mathbb{R}_0^4 \rightarrow \mathbb{CP}^1 \simeq S^2$  is a generalization of the Hopf fibration  $S^3 \rightarrow S^2$ .

Thus although we can represent the observables by means of functions, they can not be those given by Eq. (2.8). Indeed they are not projectable, since they are not homogeneous of degree zero:

$$(2.32) \quad \mathcal{L}_\Delta f_A \neq 0.$$

A possible way out is to use a conformal factor:

$$(2.33) \quad e_A(\psi) = \frac{\langle \psi | A \psi \rangle}{2 \langle \psi | \psi \rangle}$$

With this choice these functions correspond to the physical expectation value of Hermitian operators, apart from the factor  $\frac{1}{2}$ .

Notice that these functions contain the spectral information of the operator to which they are associated. Indeed, it is immediate to verify that

- the eigenvectors of the operator  $A$  correspond to the critical points of the function  $e_A$ ,
- the values of the function  $e_A$  at the critical point is precisely the eigenvalue at the corresponding eigenvector up to a factor  $\frac{1}{2}$ .

For instance, if we consider the function

$$(2.34) \quad e_3(\psi) = \frac{\langle \psi | \sigma_3 \psi \rangle}{2\langle \psi | \psi \rangle},$$

its critical points are obtained as those points  $(q^1, q^2, p_1, p_2) \in \mathbb{R}_0^4$  satisfying

$$(2.35) \quad de_3 = 0 \Rightarrow \begin{cases} q^2 = p_2 = 0 & q^1, p_1 \neq 0 \\ q^1 = p_1 = 0 & q^2, p_2 \neq 0 \end{cases}$$

The first critical set represents the eigenspace  $\text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$  while the last one represents the eigenspace  $\text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$ .

In what concerns the tensors associated to the Kähler structure of  $\mathcal{H}$  it is evident again that they can not be directly projected. Indeed, if we consider the contravariant tensors  $G$  and  $\Omega$ , we notice that they are homogeneous of degree -2:

$$(2.36) \quad \mathcal{L}_{\Delta} G = -2G; \quad \mathcal{L}_{\Delta} \Omega = -2\Omega;$$

and therefore they can not be projected onto  $\mathcal{P}$ . Instead, we can consider alternative degenerate tensor fields as

$$(2.37) \quad G_{\mathcal{P}}(\psi) = \langle \psi | \psi \rangle G(\psi) - (\Gamma \otimes \Gamma + \Delta \otimes \Delta)(\psi)$$

and

$$(2.38) \quad \Omega_{\mathcal{P}}(\psi) = \langle \psi | \psi \rangle \Omega(\psi) - (\Gamma \otimes \Delta - \Delta \otimes \Gamma)(\psi).$$

These tensor fields at  $\psi$  are clearly homogeneous of degree zero and invariant under the action of  $\Gamma$  and hence they are projectable. The factors containing  $\Gamma$  and  $\Delta$ , are chosen to make them correspond to the tensors of the canonical Kähler structure of the projective space. From the relations seen above it is immediate to verify that one-forms associated to the vector fields  $\Gamma$  and  $\Delta$  by the symplectic or the Riemannian structures are in the kernel of  $G_{\mathcal{P}}$  and  $\Omega_{\mathcal{P}}$ . Indeed, consider the mappings  $\hat{\omega} : \mathfrak{X}(\mathcal{H}) \rightarrow \Lambda^1(\mathcal{H})$

$$(2.39) \quad \hat{\omega}(X) : Y \mapsto \omega(X, Y), \quad \forall Y \in \mathfrak{X}(\mathcal{H}),$$

and analogously  $\hat{g} : \mathfrak{X}(\mathcal{H}) \rightarrow \Lambda^1(\mathcal{H})$

$$(2.40) \quad \hat{g}(X) : Y \mapsto g(X, Y), \quad \forall Y \in \mathfrak{X}(\mathcal{H}).$$

Associated with  $\Delta$  and  $\Gamma$  we have their Riemannian or symplectic dual forms, which can be seen to be related via  $-J$ :

$$(2.41) \quad \hat{\omega}(\Delta) : Y \mapsto \omega(\Delta, Y) = -\omega(\Delta, J(J(Y))) = -g(\Delta, J(Y)) = -\hat{g}(\Delta) \circ J(Y), \quad \forall Y \in \mathfrak{X}(\mathcal{H})$$

and analogously

$$(2.42) \quad \hat{\omega}(\Gamma) : Y \mapsto \omega(\Gamma, Y) = \omega(\Gamma, J(J(Y))) = -\hat{g}(\Gamma) \circ J(Y), \quad \forall Y \in \mathfrak{X}(\mathcal{H}).$$

The way  $\Gamma$  and  $\Delta$  appear in  $G_{\mathcal{P}}$  and  $\Omega_{\mathcal{P}}$  ensures that  $\hat{g}(\Delta)$  and  $\hat{g}(\Gamma)$  are in their kernels.

With respect to these tensors, we can also characterize functions  $e_A$  (Eq. (2.33)) as the functions on  $\mathcal{H}_0$  which are the pullback of functions on  $\mathcal{PH}$  such that their associated Hamiltonian fields are also Killing. We shall denote the set of these functions as  $\mathcal{E}(\mathcal{H})$ .

We can also study the gradient and Hamiltonian vector fields of the set of functions  $\mathcal{E}(\mathcal{H})$  with respect to these projectable tensors. We should consider then vector fields as

$$(2.43) \quad \mathcal{Y}_A = G_{\mathcal{P}}(de_A, \cdot); \quad e_A \in \mathcal{E}(\mathcal{H}),$$

for gradient vector fields and

$$(2.44) \quad \mathcal{X}_A = \Omega_{\mathcal{P}}(de_A, \cdot); \quad e_A \in \mathcal{E}(\mathcal{H})$$

for Hamiltonian ones.

For the example of the two level system we obtain:

$$(2.45) \quad \mathcal{Y}_{e_1} = \left( q^2 - \frac{2q^1(p_1p_2 + q^1q^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial q^1} + \left( q^1 - \frac{2q^2(p_1p_2 + q^1q^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial q^2} +$$

$$\left( p_2 - \frac{2p_1(p_1p_2 + q^1q^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial p_1} + \left( p_1 - \frac{2p_2(p_1p_2 + q^1q^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial p_2}$$

$$(2.46) \quad = Y_{f_1} - 2e_1\Delta;$$

$$(2.47) \quad \mathcal{Y}_{e_2} = \left( p_2 + \frac{2q^1(-p_2q^1 + p_1q^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial q^1} + \left( -p_1 + \frac{2q^2(-p_2q^1 + p_1q^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial q^2} +$$

$$\left( -q^2 + \frac{2p_1(-p_2q^1 + p_1q^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial p_1} + \left( q^1 - \frac{2p_2(-p_2q^1 + p_1q^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial p_2}$$

$$(2.48) \quad = Y_{f_2} - 2e_2\Delta;$$

$$(2.49) \quad \mathcal{Y}_{e_3} = \left( \frac{2q^1(p_2^2 + (q^2)^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial q^1} + \frac{1}{2} \left( \frac{-4q^2(p_1^2 + (q^1)^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial q^2} +$$

$$\left( \frac{2p_1(p_2^2 + (q^2)^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial p_1} + \left( \frac{-2p_2(p_1^2 + (q^1)^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial p_2}$$

$$(2.50) \quad = Y_{f_3} - 2e_3\Delta$$

for the gradient vector fields and

$$(2.51) \quad \mathcal{X}_{e_1} = \left( -p_2 + \frac{2p_1(p_1p_2 + q^1q^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial q^1} + \left( -p_1 + \frac{2p_2(p_1p_2 + q^1q^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial q^2} +$$

$$\left( q^2 - \frac{2q^1(p_1p_2 + q^1q^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial p_1} + \left( q^1 - \frac{2q^2(p_1p_2 + q^1q^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial p_2}$$

$$(2.52) \quad = X_{f_1} - 2e_1\Gamma;$$

$$\begin{aligned}
(2.53) \quad X_{e_2} = & \left( q^2 - \frac{2p_1(-p_2q^1 + p_1q^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial q^1} + \left( -q^1 + \frac{2p_2(-p_2q^1 + p_1q^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial q^2} + \\
& \left( p_2 - \frac{2q^1(-p_2q^1 + p_1q^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial p_1} + \left( -p_1 + \frac{2q^2(-p_2q^1 + p_1q^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial p_2}
\end{aligned}$$

$$(2.54) \quad = X_{f_2} - 2e_2\Gamma;$$

$$\begin{aligned}
(2.55) \quad X_{e_3} = & \left( \frac{-2p_1(p_2^2 + (q^2)^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial q^1} + \left( \frac{2p_2(p_1^2 + (q^1)^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial q^2} + \\
& \left( \frac{2q^1(p_2^2 + (q^2)^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial p_1} + \left( \frac{-2q^2(p_1^2 + (q^1)^2)}{(q^1)^2 + (q^2)^2 + p_1^2 + p_2^2} \right) \frac{\partial}{\partial p_2}
\end{aligned}$$

$$(2.56) \quad = X_{f_3} - 2e_3\Gamma$$

for the Hamiltonian ones.

These vector fields are tangent to the three dimensional sphere  $S^3$  of normalized vectors.

A remarkable property of these vector fields is that either one of the two families  $\{X_{f_1}, X_{f_2}, X_{f_3}\}$  or  $\{\mathcal{Y}_{f_1}, \mathcal{Y}_{f_2}, \mathcal{Y}_{f_3}\}$ , after projection, span the tangent bundle of the sphere  $S^2$ . Besides, as we saw above, they are pairwise orthogonal to each other, i.e.

$$(2.57) \quad \mathcal{Y}_{e_k} \perp X_{e_k}, \quad k = 1, 2, 3.$$

Another interesting property which we will use in the following is the fact that the Lie algebra generated by the union of both families, i.e. the Lie algebra generated by  $\{X_{f_1}, X_{f_2}, X_{f_3}, \mathcal{Y}_{f_1}, \mathcal{Y}_{f_2}, \mathcal{Y}_{f_3}\}$  is isomorphic to the special Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$  (also isomorphic to the Lie algebra of the Lorentz group). If we consider the algebra generated by the vector fields defined on  $\mathcal{H}$ , we see that  $\{X_{f_1}, X_{f_2}, X_{f_3}, Y_{f_1}, Y_{f_2}, Y_{f_3}\}$  generate again the Lie algebra  $\mathfrak{sl}(2, \mathbb{C})$ . If we include the vector fields  $Y_{f_0} \propto \Delta$  and  $X_{f_0} \propto \Gamma$ , the set generates the full  $\mathfrak{gl}(n, \mathbb{C})$ .

As a curiosity we can study the flow associated to these fields. For instance, if we consider vector fields  $X_{e_3}$  and  $\mathcal{Y}_{e_3}$  and represent their flows from an initial condition  $\psi_0$  we obtain the flows presented in Figures 1 and 3.

In Figure 1 we can see how the flow takes us towards one of the eigenspaces of the operator  $\sigma_3$ , in particular the one generated by  $\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ . In this particular numerical example the limit point is  $q^1 = 0.5547$  and  $p_1 = 0.83205$ , with the remaining two coordinates vanishing, i.e., the system is selecting the eigenvector  $\psi = \begin{pmatrix} 0.5547 + i0.83205 \\ 0 \end{pmatrix}$ . But this point is obviously connected by the flow of  $\Gamma$  with the point  $(1, 0, 0, 0)$ , as it can be seen in Figure 2. This is just reflecting that both points are on an integral curve of  $\Gamma$ .

The flow of the Hamiltonian vector field depicted in Fig 3 is completely different. We see how the evolution of all four coordinates is periodic, with two different frequencies, one for  $q^1, p_1$  and the other for  $q^2, p_2$ . We can also verify in Figure 4 that the flow is projecting onto the projective space by checking that the flow commutes with the flows of the vector fields  $\Gamma$  and  $\Delta$ .

**2.1.3. Alternative structures and representations of the unitary group.** We notice that the relevant tensor fields associated with the geometry of the projective space are invariant under the unitary group. An important comment is however in order. The specific form of the Hermitian structure (2.2) which selects the realization of the unitary group acting



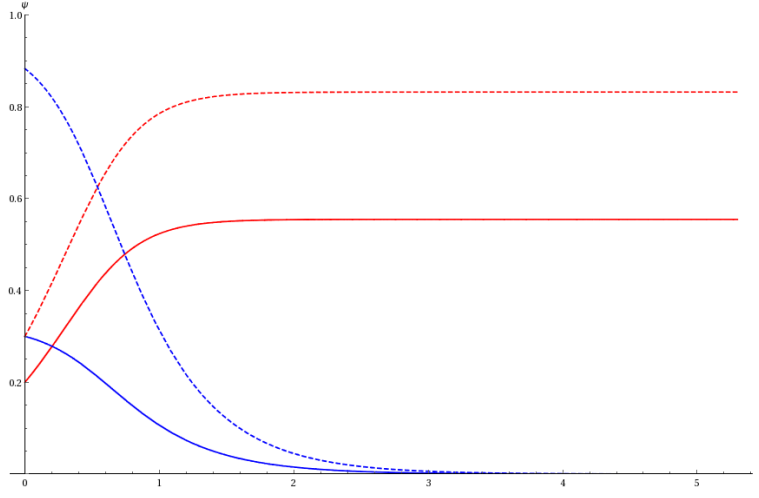


FIGURE 1. Flow of the gradient vector field  $\mathcal{Y}_{e_3}$  from the point on the unit sphere with coordinates  $q^1 = 0.2$ ,  $q^2 = 0.3$ ,  $p_1 = 0.3$ . The red solid line is the flow for  $q^1$ , the dashed red line  $p_1$ , the blue solid line  $q^2$  and the blue dashed line is  $p_2$ .

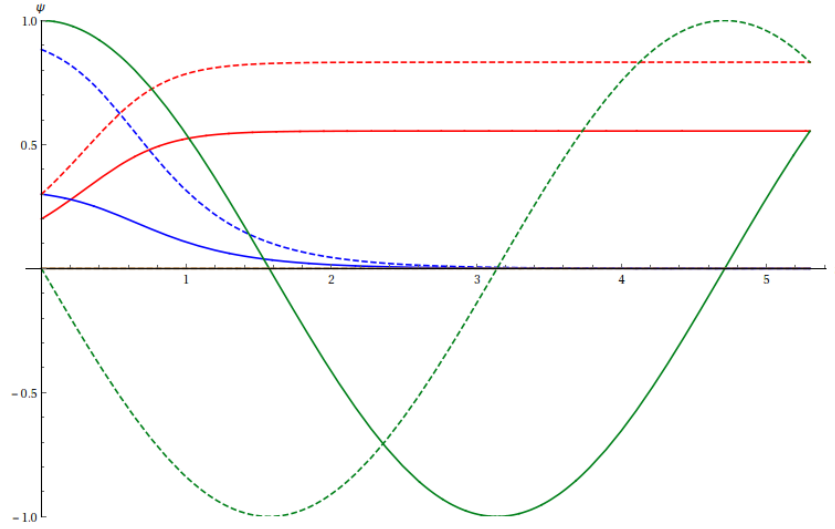


FIGURE 2. Flow of the gradient vector field  $\mathcal{Y}_{e_3}$  from the point on the unit sphere with coordinates  $q^1 = 0.2$ ,  $q^2 = 0.3$  and  $p_1 = 0.3$  and flow of  $\Gamma$  (in green) from the point  $(1, 0, 0, 0)$ . We can see how both flows coincide at one point, thus proving that the limit point of the gradient flow is in the same complex ray as  $(1, 0, 0, 0)$ .

on the Hilbert space in the defining representation is not the only possible one. Eq. 2.2 represents the Hermitian structure of the abstract Hilbert space once we have chosen a basis which defines a one-to-one correspondence between the Hilbert space  $\mathcal{H}$  and  $\mathbb{C}^2$ . But we could also consider a different realization of the Hermitian structure, say in the same basis.

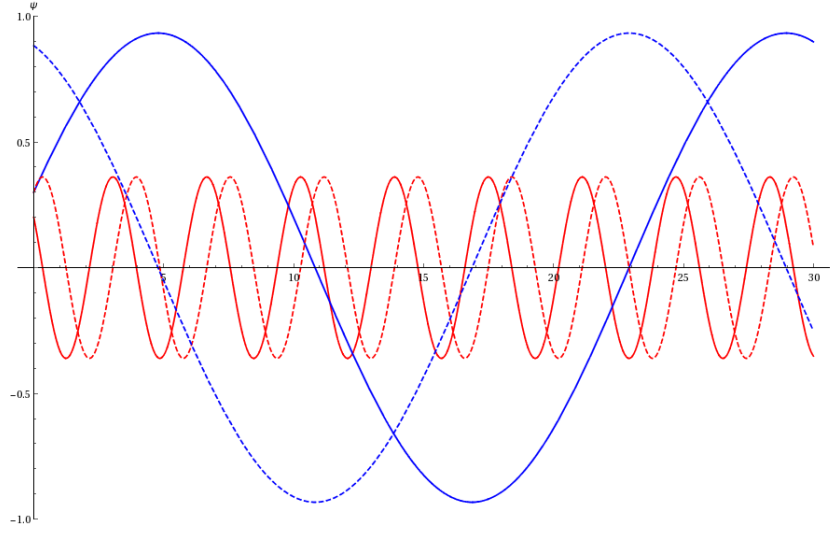


FIGURE 3. Flow of the Hamiltonian vector field  $X_{e_3}$  from the point on the unit sphere with coordinates  $q^1 = 0.2$ ,  $q^2 = 0.3$ ,  $p_1 = 0.3$ . The red solid line is the flow for  $q^1$ , the dashed red line  $p_1$ , the blue solid line  $q^2$  and the blue dashed line is  $p_2$ .

Consider, for instance, a scalar product such as:

$$(2.58) \quad \left\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \right\rangle_\alpha = \alpha \bar{z}_1 w_1 + (2 - \alpha) \bar{z}_2 w_2; \quad 0 < \alpha \leq 1$$

This operation defines clearly a complex scalar product on  $\mathbb{C}^2$  and defines an alternative Hilbert space structure. With it we obtain a different realization of the unitary group as the isometry group of the new structure, this new realization possesses all the “abstract” properties which characterize the unitary group. At the level of the Lie algebra realization, we would have the vector fields associated with those matrices which are Hermitian with respect to the matrix representation

$$(2.59) \quad K = \begin{pmatrix} \alpha & 0 \\ 0 & 1 - \alpha \end{pmatrix},$$

i.e., those matrices  $A$  satisfying

$$(2.60) \quad A^\dagger K A = K.$$

The point we would like to emphasize is that the “geometry” which a given group identifies is associated with the “abstract group” and not with a specific realization.

## 2.2. The Heisenberg picture.

2.2.1. *The algebraic formulation.* In a modern language, Heisenberg picture uses as the relevant carrier space to describe a quantum system a  $C^*$ -algebra  $\mathcal{A}$ .

**Definition 1.** A  $C^*$ -algebra is a Banach algebra over  $\mathbb{C}$  with a map  $*$  :  $\mathcal{A} \rightarrow \mathcal{A}$  satisfying

- $*$  is an involution, i.e.,  $((a^*)^*)^* = a$  for all  $a \in \mathcal{A}$
- the effect of the involution on the algebra structure is given as
  - $(a + b)^* = a^* + b^*$  for any  $a, b \in \mathcal{A}$ ,

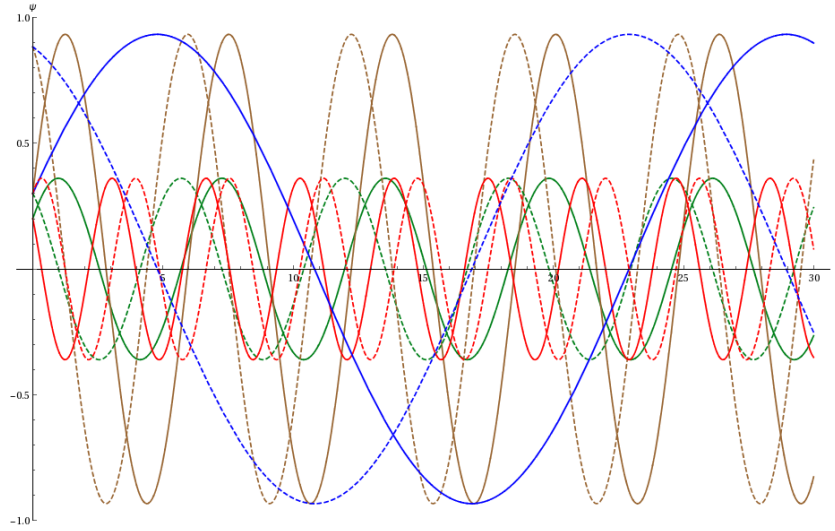


FIGURE 4. Flow of the Hamiltonian vector field  $X_{e_3}$  and the vector field  $\Gamma$  from the point on the unit sphere with coordinates  $q^1 = 0.2$ ,  $q^2 = 0.3$ ,  $p_1 = 0.3$ . The red and blue lines represent the flow of  $X_{e_3}$ , while the green and brown lines represent the flow of  $\Gamma$ . It is simple to verify that there are no common points, i.e. solid red and solid green lines do not intersect at the same time that dashed red and green, solid blue and brown and dashed blue and brown.

- $(ab)^* = b^*a^*$  for any  $a, b \in \mathcal{A}$ ,
- $(\lambda a)^* = \bar{\lambda}a^*$  for any  $\lambda \in \mathbb{C}$  ( $\bar{\lambda}$  represents the complex conjugate) and any  $a \in \mathcal{A}$ ,
- $\|aa^*\| = \|a\|\|a^*\|$  for any  $a \in \mathcal{A}$ .

On this carrier space, states correspond to positive and normalized linear functionals  $\rho$  and observables to the real subalgebra corresponding to the elements  $A \in \mathcal{A}$  which are stable under the  $*$ -operation. These correspond to the real elements of the algebra.

In the case of the two level system which we studied above, the  $C^*$  algebra  $\mathcal{A}$  corresponds to the  $2 \times 2$  complex matrices, i.e.  $\mathcal{A} = M_2(\mathbb{C})$ , with respect to the usual associative product. The  $*$  operation corresponds to the adjoint  $A \mapsto A^\dagger$  and the real elements correspond to the self-adjoint matrices  $A^\dagger = A$ . This set,  $\text{Herm}(\mathcal{H})$ , is then isomorphic to the Lie algebra of the unitary group  $\mathfrak{u}(2) \subset M_2(\mathbb{C}) \simeq \mathfrak{gl}(2, \mathbb{C})$ , the isomorphism being just a multiplication by the imaginary unit (since the Lie algebra  $\mathfrak{u}(2)$  is constituted by anti-Hermitian operators). The norm required to make  $\mathcal{A}$  into a Banach algebra can be given by the trace  $\|A\|^2 = \text{Tr}(A^\dagger A)$  which comes from the scalar product on the Lie algebra  $\mathfrak{gl}(\mathcal{H})$   $\langle A|B \rangle = \text{Tr}(A^\dagger B)$  and hence on the unitary algebra. It is immediate to verify that these objects satisfy the conditions above.

To make comparison easier with the Schrödinger picture, we can consider a description in terms of a Lie-Jordan(-Banach) algebra  $\mathcal{L}$  whose complexification  $\mathcal{L}^\mathbb{C}$  is isomorphic to  $\mathcal{A}$ . We define thus

**Definition 2.** A vector space  $\mathcal{L}$  endowed with a Jordan algebra structure  $\circ$  and a Lie structure  $[\cdot, \cdot]$ , such that  $\forall a, b, c \in \mathcal{L}$ :

- the Lie-adjoint define derivations of the Jordan operation, i.e,  $[a, b \circ c] = [a, b] \circ c + b \circ [a, c]$
- the associators of the two operations are proportional to each other  $(a \circ b) \circ c - a \circ (b \circ c) = \hbar^2 ([a, b], c) - [a, [b, c]]$  where  $\hbar \in \mathbb{R}$ ,

is called a **Lie-Jordan algebra**.

**Definition 3.** A Lie-Jordan algebra  $\mathcal{L}$  endowed with a norm  $\|\cdot\|$  such that  $\mathcal{L}$  is complete and satisfies

- $\|a \circ b\| \leq \|a\| \|b\|$
- $\|[a, b]\| \leq |\hbar|^{-1} \|a\| \|b\|$
- $\|a^2\| = \|a\|^2$
- $\|a^2\| \leq \|a^2 + b^2\|$

for any  $a, b \in \mathcal{L}$ , is called a **Lie-Jordan-Banach (LJB) algebra**.

In the case of a  $n$ -level quantum system, the LJB algebra  $\mathcal{L}$  becomes the set of Hermitian operators  $\text{Herm}(\mathcal{H})$ , which is isomorphic to the unitary algebra  $\mathfrak{u}(n)$ , which can be also identified with its dual  $\mathfrak{u}^*(n)$ . From the practical point of view, we shall consider all our objects defined on this vector space. The two products  $[\cdot, \cdot]$  and  $\circ$  arise as the skew-symmetric and symmetric part (respectively) of the associative product defining the  $C^*$ -algebra structure of  $\mathcal{A}$ .

**2.2.2. Geometrical formalism and the momentum map.** One important convention we shall use in the following is the identification of the dual space  $\mathfrak{u}^*(\mathcal{H})$  with the set of Hermitian operators. Thus, given any  $\xi^\dagger = \xi$  and  $T^\dagger = -T$ ,

$$(2.61) \quad \langle \xi | T \rangle = \frac{i}{2} \text{Tr}(\xi T).$$

Notice that under this identification  $\mathfrak{u}^*(\mathcal{H})$  becomes a Lie algebra with the bracket

$$(2.62) \quad [\xi_1, \xi_2] := -i(\xi_1 \xi_2 - \xi_2 \xi_1), \quad \forall \xi_1, \xi_2 \in \mathfrak{u}^*(\mathcal{H}),$$

which is isomorphic to  $\mathfrak{u}(\mathcal{H})$ , by the isomorphism

$$(2.63) \quad \mathfrak{u}^*(\mathcal{H}) \ni \xi \mapsto \hat{\xi} = -i\xi \in \mathfrak{u}(\mathcal{H}).$$

Analogously, we can also export the scalar product from  $\mathfrak{u}(\mathcal{H})$  to  $\mathfrak{u}^*(\mathcal{H})$  and define

$$(2.64) \quad \langle \xi_1, \xi_2 \rangle_{\mathfrak{u}^*} = \frac{1}{2} \text{Tr}(\xi_1 \xi_2) \quad \forall \xi_1, \xi_2 \in \mathfrak{u}^*(\mathcal{H}).$$

This scalar product allows us to identify linear functionals on  $\mathfrak{u}^*(\mathcal{H})$  as elements on the Lie algebra  $\mathfrak{u}(\mathcal{H})$ , recovering the isomorphism (2.63):

$$(2.65) \quad \mathfrak{u}^*(\mathcal{H}) \ni A \rightarrow \hat{A} := \langle A, \cdot \rangle_{\mathfrak{u}^*} = -iA \in \mathfrak{u}(\mathcal{H})$$

From the tensorial point of view, it is possible to encode the symmetric and skew-symmetric products of Hermitian matrices in two tensors  $R$  and  $\Lambda$  defined on the dual space  $\mathfrak{u}^*(n)$  as

$$(2.66) \quad R(\xi)(d\hat{A}, d\hat{B}) = \langle \xi | A \circ B \rangle_{\mathfrak{u}^*}, \quad \forall \xi \in \mathfrak{u}^*(\mathcal{H}), A, B \in \text{Herm}(\mathcal{H})$$

and

$$(2.67) \quad \Lambda(\xi)(d\hat{A}, d\hat{B}) = \langle \xi | [A, B] \rangle_{\mathfrak{u}^*}, \quad \forall \xi \in \mathfrak{u}^*(\mathcal{H}), A, B \in \text{Herm}(\mathcal{H}).$$

Tensor  $R$  encodes the Jordan structure while  $\Lambda$  corresponds to the Lie structure on the dual of the unitary algebra. The dynamics, corresponding to Heisenberg's equation, arises

as the integral curve of the vector field  $X_{\hat{H}} \in \mathfrak{X}(\mathfrak{u}^*(\mathcal{H}))$  which is the Hamiltonian vector field corresponding to the operator  $H$ :

$$(2.68) \quad X_{\hat{H}} = \Lambda(d\hat{H}, \cdot).$$

Integral curves are the orbits of a one parameter subgroup in the co-adjoint action of the unitary group on  $\mathfrak{u}^*(\mathcal{H})$ . This property shall be used in the next Section.

It is possible to relate these tensors with those defined on  $\mathcal{H}$  and  $\mathcal{P}$  which encode the Schrödinger formalism. The first relevant aspect is the defining action of the unitary group on the Hilbert space  $\mathcal{H}$  and the corresponding projective space  $\mathcal{P}$ :

$$(2.69) \quad \Phi : U(\mathcal{H}) \times \mathcal{H} \rightarrow \mathcal{H}; \quad \Phi_{\mathcal{P}} : U(\mathcal{H}) \times \mathcal{P} \rightarrow \mathcal{P}$$

By definition,  $\Phi$  acts as isometries of the Hermitian structure of  $\mathcal{H}$ , and therefore, preserves the Kähler structure  $(g, \omega, J)$ . Hence the action  $\Phi$  is symplectic and must admit a momentum map. The same happens with  $\Phi_{\mathcal{P}}$ . Thus we have two projections

$$(2.70) \quad \mu : \mathcal{H} \rightarrow \mathfrak{u}^*(\mathcal{H}); \quad \psi \mapsto P_{\psi} = \frac{1}{2}|\psi\rangle\langle\psi|$$

and

$$(2.71) \quad \mu_{\mathcal{P}} : \mathcal{P} \rightarrow \mathfrak{u}^*(\mathcal{H}); \quad [\psi] \mapsto \rho_{\psi} = \frac{|\psi\rangle\langle\psi|}{2\langle\psi|\psi\rangle}$$

The fundamental vector fields of these actions correspond to the Hamiltonian vector fields  $X_{f_A}$  on  $\mathcal{H}$  and  $X_{e_A}$  on  $\mathcal{P}$ , for any  $A \in \text{Herm}(\mathcal{H})$ . We know that we can associate linear functions on  $\mathfrak{u}^*(\mathcal{H})$  to operators in  $\text{Herm}(\mathcal{H})$  (or equivalently on  $\mathfrak{u}(\mathcal{H})$ ) as

$$(2.72) \quad \text{Herm}(\mathcal{H}) \ni A \mapsto \langle P_{\psi}|A\rangle_{\mathfrak{u}^*} = F_A(P_{\psi})$$

and

$$(2.73) \quad \text{Herm}(\mathcal{H}) \ni A \mapsto \left\langle \rho_{\psi} \middle| A \right\rangle_{\mathfrak{u}^*} = E_A(\rho_{\psi}).$$

The pullback of these functions by  $\mu$  and  $\mu_{\mathcal{P}}$  correspond to the functions on  $\mathcal{H}$  (resp.  $\mathcal{P}$ )

$$(2.74) \quad \mu^*(F_A(P_{\psi})) = f_A(\psi) = \frac{1}{2}\langle\psi|A\psi\rangle$$

and

$$(2.75) \quad \mu_{\mathcal{P}}^*(E_A(\rho_{\psi})) = e_A(\psi) = \frac{\langle\psi|A\psi\rangle}{2\langle\psi|\psi\rangle}.$$

These simple relations allow us to verify that the tensors  $G$  and  $\Omega$  on  $\mathcal{H}$  (respectively  $G_{\mathcal{P}}$  and  $\Omega_{\mathcal{P}}$  for the projective case) are  $\mu$  (respectively  $\mu_{\mathcal{P}}$ ) related to tensors  $R$  and  $\Lambda$  defined on  $\mathfrak{u}^*(\mathcal{H})$ . Indeed, if we compute the action of the tensors  $G$  or  $\Omega$  on two pullback functions  $f_A(\psi) = \mu^*(F_A(P_{\psi}))$  and  $f_B(\psi) = \mu^*(F_B(P_{\psi}))$  for two arbitrary Hermitian operators  $A, B \in \text{Herm}(\mathcal{H})$  we obtain:

$$(2.76) \quad G(df_A, df_B)(\psi) = f_{A \circ B}(\psi) = \mu^*(F_{A \circ B}(P_{\psi})) = \mu^*(\langle P_{\psi}|A \circ B\rangle_{\mathfrak{u}^*}) = \mu^*(R(d\hat{A}, d\hat{B})(P_{\psi}))$$

Analogously

$$(2.77) \quad \Omega(df_A, df_B)(\psi) = f_{[A, B]}(\psi) = \mu^*(F_{[A, B]}(P_{\psi})) = \mu^*(\langle P_{\psi}|[A, B]\rangle_{\mathfrak{u}^*}) = \mu^*(\Lambda(d\hat{A}, d\hat{B})(P_{\psi})).$$

In an analogous way we can see how tensors  $R$  and  $\Lambda$  are  $\mu_{\mathcal{P}}$ -related to tensors  $G_{\mathcal{P}}$  and  $\Omega_{\mathcal{P}}$ :

$$\begin{aligned}
(2.78) \quad \mu_{\mathcal{P}}^* \left( \Lambda(d\hat{A}, d\hat{B})(\rho_{\psi}) \right) &= \mu_{\mathcal{P}}^* \left( \left\langle \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} [A, B] \right\rangle_{\mathfrak{u}^*} \right) \\
&= \mu_{\mathcal{P}}^* \left( \frac{1}{2} \text{Tr} \left( \frac{|\psi\rangle\langle\psi|}{\langle\psi|\psi\rangle} [A, B] \right) \right) = \mu_{\mathcal{P}}^* \left( E_{[A, B]}(\rho_{\psi}) \right) = e_{[A, B]}(\psi) = \Omega_{\mathcal{P}}(\psi)(de_A, de_B).
\end{aligned}$$

The symmetric tensor is slightly different because of the projective nature. In any case, it is also straightforward that

$$(2.79) \quad \mu_{\mathcal{P}}^* \left( R(d\hat{A}, d\hat{B})(\rho_{\psi}) \right) = \mu_{\mathcal{P}}^* (E_{[A \circ B]}(\rho_{\psi})) = e_{A \circ B}(\psi) = G_{\mathcal{P}}(de_A, de_B)(\psi) + e_A(\psi)e_B(\psi).$$

These relations also determine the equivalence of Schrödinger's and Heisenberg's pictures when both are meaningful. Indeed, because of the correspondences between the operators and the functions, it is immediate to verify that the Hamiltonian vector field  $X_{f_h} = -\Omega(df_H, \cdot)$  whose integral curves correspond to the solutions of Schrödinger equation is mapped by  $T\mu : T\mathcal{H} \rightarrow T\mathfrak{u}^*(\mathcal{H})$  onto the Hamiltonian vector field  $X_{\hat{H}} = -\Lambda(d\hat{H}, \cdot)$  whose integral curves define the solutions of Heisenberg equation on  $\mathfrak{u}^*(\mathcal{H})$ :

$$(2.80) \quad T\mu(X_{f_h}) = X_{\hat{H}}.$$

Analogously, if we consider the momentum mapping for  $\mathcal{P}$ ,  $T\mu_{\mathcal{P}} : T\mathcal{P} \rightarrow T\mathfrak{u}^*(\mathcal{H})$ , we obtain that the image of the solutions of the projective Schrödinger equation define also the integral curves of Heisenberg equation:

$$(2.81) \quad T\mu_{\mathcal{P}}(X_{e_H}) = X_{\hat{H}}.$$

Notice that each momentum map has a different image. We can proceed analogously with any Hamiltonian vector field on both spaces. We can also map the gradient vector fields.

If we consider again our example  $\mathcal{H} = \mathbb{C}^2$ , we identify:

- the image of the Hamiltonian vector fields:

$$(2.82) \quad T\mu(X_{f_k}) = X_{\hat{\sigma}_k}; \quad k = 1, 2, 3,$$

- the image of the gradient vector fields

$$(2.83) \quad T\mu(Y_{f_k}) = Y_{\hat{\sigma}_k} := R(d\hat{\sigma}_k, \cdot); \quad k = 1, 2, 3,$$

- the image of the vector field  $\Delta$ , which is the gradient vector field associated to the norm:

$$(2.84) \quad T\mu(\Delta) = Y_{\hat{\sigma}_0} = R(d\hat{\sigma}_0, \cdot),$$

and which is proportional (but not equal) to the natural dilation vector field  $(\Delta_{\mathfrak{u}^*})$  of the linear structure of  $\mathfrak{u}^*(\mathcal{H})$ . The precise relation is

$$(2.85) \quad T\mu(\Delta_{\mathcal{H}}) = 2\Delta_{\mathfrak{u}^*}.$$

The image of the vector field  $\Gamma$  which is the Hamiltonian vector field corresponding to the norm vanishes because of the skew-symmetry of the tensor:

$$(2.86) \quad T\mu(\Gamma) = X_{\hat{\sigma}_0} = \Lambda(d\hat{\sigma}_0, \cdot) = 0.$$

If we consider the vector fields associated to the Hamiltonian function in both spaces we see immediately how the vector field associated to the Schrödinger equation on  $\mathcal{H}$  (i.e.,  $X_{f_h}$ ) is mapped onto the vector field associated to von Neumann equation on  $\mathfrak{u}^*(\mathcal{H})$ , i.e.  $X_{\hat{H}}$ .

The set of vector fields  $\{X_{\hat{\sigma}_\mu}, Y_{\hat{\sigma}_\mu}\}$  with  $\mu = 0, 1, 2, 3$  do not generate now the Lie algebra  $\mathfrak{gl}(\mathcal{H}, \mathbb{C})$ , since the Hamiltonian vector field associated to the identity operator is identically

zero. This implies that on the image  $\mu(\mathcal{H}) \subset \mathfrak{u}^*(\mathcal{H})$  we can recover the action of the group  $GL(\mathcal{H}, \mathbb{C})$  which we found at the level of the Hilbert space, excepting the global phase change of the determinant. Notice that if we consider only the Hamiltonian vector fields  $\{X_{\hat{\sigma}_\mu}\}$  with  $\mu = 0, 1, 2, 3$  they generate the Lie algebra of the special unitary group  $SU(\mathcal{H})$ , they obviously define an involutive distribution. The corresponding foliation defines the set of orbits of the coadjoint action of the unitary group on the dual of its Lie algebra. We can parametrize the points in  $\mathfrak{u}^*(\mathcal{H})$  by coordinates  $\{y^0, y^1, y^2, y^3\}$  corresponding to

$$(2.87) \quad \mathfrak{u}^*(\mathcal{H}) \ni \rho = \sum_k y^k \sigma_k; \quad y^k(\rho) = \langle \sigma_k | \rho \rangle_{\mathfrak{u}^*} = \frac{1}{2} \text{Tr}(\sigma_k \rho).$$

The leaves of the foliation will be two dimensional spheres.

If we ask the trace of the state  $\rho \in \mathfrak{u}^*(\mathcal{H})$  to be equal to one,  $y^0 = \frac{1}{2}$ . The set of physical states correspond then to the points which are contained in the three dimensional ball

$$(2.88) \quad \mathcal{D}(\mathcal{H}) = \left\{ \rho \in \mathfrak{u}^*(\mathcal{H}) \mid y^0 = \frac{1}{2}; \quad (y^1)^2 + (y^2)^2 + (y^3)^2 \leq \frac{3}{4} \right\}$$

The distribution generated by  $\{X_{\hat{\sigma}_\mu}\}$  is tangent to any sphere contained in  $\mathcal{D}(\mathcal{H})$ . Instead, if we consider the distribution generated by the gradient vector fields, we can verify immediately that it is not involutive, since the commutator of two gradient vector fields is a Hamiltonian vector field and therefore the operation is not inner. Besides, the distribution generated by the gradient vector fields is tangent to the surface of the outmost sphere in  $\mathcal{D}(\mathcal{H})$ , but not to the spheres in the interior of the ball. With respect to them, in general it will be transversal.

The situation is different if we consider the projection  $\mu_{\mathcal{P}}$ . Indeed, it is immediate to notice that vector fields  $\Delta$  and  $\Gamma$  are annihilated by the projection  $\pi : \mathcal{H} \rightarrow \mathcal{P}$  and hence on the image by  $\mu_{\mathcal{P}}$  there are only six generators  $\{X_{\hat{\sigma}_k}, Y_{\hat{\sigma}_k}\}$  for  $k = 1, 2, 3$  and where

$$T\mu_{\mathcal{P}}(X_{e_k}) = X_{\hat{\sigma}_k}, \quad T\mu_{\mathcal{P}}(Y_{e_k}) = Y_{\hat{\sigma}_k}$$

Therefore on the submanifold  $\mu_{\mathcal{P}}(\mathcal{P}) \subset \mathfrak{u}^*(\mathcal{H})$  we can consider only the action of the group  $SL(\mathcal{H}, \mathbb{C})$ , whose Lie algebra is generated by the vector fields  $\{X_{\hat{\sigma}_k}, Y_{\hat{\sigma}_k}\}$  for  $k = 1, 2, 3$ . Analogously, when considering the integrable distribution generated by the Hamiltonian vector fields  $\{T\mu_{\mathcal{P}}(X_{e_k})\}$  for  $k = 1, 2, 3$ , we identify the orbit of the special unitary group  $SU(2)$ .

This change in the global group to  $SL(\mathcal{H})$  when considering the complex projective space (both as a manifold or as its image by  $\mu_{\mathcal{P}}$ ) is also reflecting a quite remarkable property, namely the nonlocality of the product  $\star$  which encodes the associative product of operators. Indeed, we already explained that it is via this product how we can build a  $C^*$ -algebra structure on the space of functions  $\mathcal{E}(\mathcal{H})$ . Thus we can write

$$(2.89) \quad e_{AB} := e_A \star e_B = e_A e_B + \frac{1}{2} G_{\mathcal{P}}(de_A, de_B) + \frac{i}{2} \Omega_{\mathcal{P}}(de_A, de_B),$$

where we must keep in mind that the product is associative and non-local. Thus  $(\mathcal{E}_{\mathbb{C}}(\mathcal{H}), \star)$  (i.e., the complexification of the algebra of functions generated by Hermitian operators) becomes a  $C^*$ -algebra. Now, if we consider the automorphisms of this algebra, we will identify the whole group  $GL(\mathcal{H})$  acting on  $\mathcal{E}_{\mathbb{C}}(\mathcal{H})$  as automorphisms with respect to the  $\star$  operation, i.e., transformations of the type

$$(2.90) \quad \Phi_T : e_A \mapsto e_T \star e_A \star e_{T^{-1}} = e_{TAT^{-1}}, \quad T \in GL(\mathcal{H}).$$

As this product is non-local, transformations on functions do not induce transformations on the space on which they are defined. In other terms, infinitesimal generators of one

parameter subgroups will not be derivations of the pointwise product of functions, therefore we will not have vector fields associated to them as infinitesimal generators.

**2.2.3. The GNS construction.** We saw in the previous section how the momentum mapping allows us to map the Schrödinger picture on the Heisenberg one. Let us see now how the GNS construction allows us to move in the other direction, i.e., from the Heisenberg picture we will recover the Schrödinger one.

The starting point is thus a  $C^*$ -algebra  $\mathcal{A}$  which contains the set of physical observables as a real subspace. States are represented as normalized positive linear functionals  $\omega$  which satisfy

$$(2.91) \quad \omega(a^*a) \geq 0; \quad \omega(\mathbb{I}) = 1.$$

Therefore, we can embed the set of states  $\mathcal{D}(\mathcal{A})$  in the dual of  $\mathcal{A}$ .

Each state  $\omega$  allows us to introduce a pairing between the elements of  $\mathcal{A}$ :

$$(2.92) \quad \langle a, b \rangle_\omega = \omega(a^*b).$$

The pairing is positive because of the properties of  $\omega$  but it may be degenerate. We define then the Gelfand ideal  $\mathcal{I}_\omega$  to be the kernel:

$$(2.93) \quad \mathcal{I}_\omega = \{a \in \mathcal{A} \mid \omega(a^*a) = 0\}$$

If we define the quotient space  $\tilde{\mathcal{H}}_\omega = \mathcal{A}/\mathcal{I}_\omega$  and the subsequent equivalence classes

$$(2.94) \quad \Psi_a = \{a + \alpha \mid a \in \mathcal{A}; \alpha \in \mathcal{I}_\omega\},$$

we can define a structure of pre-Hilbert space on  $\tilde{\mathcal{H}}_\omega$  by using the scalar product

$$(2.95) \quad \langle \Psi_a | \Psi_b \rangle = \omega(a^*b).$$

By completing  $\tilde{\mathcal{H}}_\omega$  with respect to the corresponding norm topology, we define a Hilbert space  $\mathcal{H}_\omega$ .

On this Hilbert space we can define a representation of the  $C^*$ -algebra  $\mathcal{A}$  in the form:

$$(2.96) \quad \pi_\omega : \mathcal{A} \times \mathcal{H}_\omega \rightarrow \mathcal{H}_\omega; \quad \pi_\omega(b)\Psi_a = \Psi_{ab}.$$

In more formal terms we have

**Definition 4.** A  $*$ -representation of  $\mathcal{A}$  on the Hilbert space  $\mathcal{H}_\omega$  is a homomorphism  $\pi_\omega$  from  $\mathcal{A}$  to the algebra  $\mathcal{B}(\mathcal{H}_\omega)$  of bounded operators on  $\mathcal{H}_\omega$  which maps the involution of  $\mathcal{A}$  on the adjoint operation of  $\mathcal{B}(\mathcal{H}_\omega)$ .

If we consider the vector associated to the identity element of  $\mathcal{A}$ , i.e.

$$|\Omega\rangle = \Psi_{\mathbb{I}},$$

we can recover the state  $\omega$  as

$$(2.97) \quad \omega(a) = \langle \Omega | \pi_\omega(a) \Omega \rangle, \quad \forall a \in \mathcal{A}.$$

As we know that the set of states  $\mathcal{D}(\mathcal{A})$  can be embedded in  $\mathcal{A}$ , the expression above implies that the Hilbert space  $\mathcal{H}_\omega$  may be thought of as a subspace of  $\mathcal{A}$  coinciding with the orbit of the left action of  $\mathcal{A}$  on itself which passes through  $\omega$ , once identified with an element of  $\mathcal{A}$  (remember that we are in finite dimension):

$$(2.98) \quad \mathcal{H}_\omega \simeq \mathcal{O}_{\mathcal{A}}(\omega).$$

This can also be done for any other element of  $\mathcal{H}_\omega$ :



**Definition 5.** Given  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{H}_\omega)$ , a vector  $\xi \in \mathcal{H}_\omega$  is said to be **cyclic** if the orbit

$$\mathcal{O}_\xi = \{\pi_\omega(x)\xi | x \in \mathcal{A}\}$$

is dense in  $\mathcal{H}_\omega$ . If a cyclic vector exists, we say that  $\pi_\omega$  is a **cyclic representation**. If we consider a normalized vector  $\|\xi\| = 1$ , the functional

$$\rho_\xi : x \mapsto \langle \xi | \pi_\omega(x) \xi \rangle \in \mathbb{C}$$

is a state of  $\mathcal{A}$ .

In this context we notice that the ambiguity in the Hermitian structure of the corresponding Hilbert space is connected with the ambiguity in the choice of the starting fiducial state to define the GNS Hilbert space and the subsequent realization of the unitary group. The abstract group is always the same (the unitary group) but the realizations may be different.

Besides, from the state  $\omega$  (i.e., from  $|\Omega\rangle$ ) we can define other states as density matrices  $\rho$  defined on  $\mathcal{B}(\mathcal{H}_\omega)$  as the convex combinations of projectors on one dimensional subspaces of  $\mathcal{H}_\omega$ :

$$(2.99) \quad \omega_\rho(a) = \text{Tr}(\rho \pi_\omega(a)).$$

These generalized states lead to representations of  $\mathcal{A}$  which are reducible and decomposable as a direct sum:

$$(2.100) \quad \pi_\rho = \bigoplus_\alpha \pi_\alpha$$

on subspaces

$$(2.101) \quad \mathcal{H}_\rho = \bigoplus_\alpha \mathcal{H}_\alpha$$

The vacuum state  $|\Omega_\rho\rangle$  corresponding to the identity element of  $\mathcal{A}$  decomposes then as a sum:

$$(2.102) \quad |\Omega_\rho\rangle = \sum_\alpha |\Omega_\alpha\rangle; \quad |\Omega_\alpha\rangle \in \mathcal{H}_\alpha,$$

in such a way that the irreducible representations  $\pi_\alpha$  are associated to pure states  $\xi_\alpha$ :

$$(2.103) \quad \xi_\alpha(a) = \frac{1}{p_\alpha} \langle \Omega_\alpha | \pi_\alpha(a) | \Omega_\alpha \rangle,$$

where  $p_\alpha = \langle \Omega_\alpha | \Omega_\alpha \rangle$ . As  $|\Omega_\rho\rangle$  is normalized,  $\sum_\alpha p_\alpha = 1$ . But then we can write the state  $\rho$  as a convex combination

$$(2.104) \quad \rho = \sum_\alpha p_\alpha \xi_\alpha,$$

where  $\{\xi_\alpha\}$  are pure states.

Notice that, from a geometric point of view, once we have fixed a state  $\omega$ , we can reproduce the analysis of the momentum map which we considered in the previous section at the level of the Hilbert space  $\mathcal{H}_\omega$ . Indeed, we can consider the unitary group  $U(\mathcal{H}_\omega)$  and its defining action on  $\mathcal{H}_\omega$ :

$$(2.105) \quad \Phi : U(\mathcal{H}_\omega) \times \mathcal{H}_\omega \rightarrow \mathcal{H}_\omega; \quad \Psi_a \mapsto U\Psi_a,$$

and on the corresponding projective space

$$(2.106) \quad \Phi_{\mathcal{P}} : U(\mathcal{H}_\omega) \times \mathcal{P}\mathcal{H}_\omega \rightarrow \mathcal{P}\mathcal{H}_\omega; \quad [\Psi_a] \mapsto [U\Psi_a].$$

Associated to these actions we can define two projections in analogy to what we did in the previous section:

$$(2.107) \quad \mu_\omega : \mathcal{H}_\omega \rightarrow \mathfrak{u}^*(\mathcal{H}_\omega); \quad \mu_\omega(\Psi_a) = \frac{1}{2}|\Psi_a\rangle\langle\Psi_a|,$$

and

$$(2.108) \quad \mu_\omega^\mathcal{P} : \mathcal{P}\mathcal{H}_\omega \rightarrow \mathfrak{u}^*(\mathcal{H}_\omega); \quad \mu_\omega^\mathcal{P}([\Psi_a]) = \frac{|\Psi_a\rangle\langle\Psi_a|}{2\langle\Psi_a|\Psi_a\rangle}.$$

$\mu_\omega$  defines (as in the case of the previous section) a symplectic realization on the symplectic vector space  $\mathcal{H}_\omega$  of the Poisson manifold  $\mathfrak{u}^*(\mathcal{H}_\omega)$ . As  $\mathcal{H}_\omega$  is a vector space, it is called a Hilbert space realization.  $\mu_\omega^\mathcal{P}$  also defines a symplectic realization, but in this case we call it a Kählerian realization.

**2.2.4. Further comments.** Within the study of the  $C^*$ -algebraic approach we can identify transformation groups given by:

- automorphisms of the Jordan structure;
- automorphisms of the Lie structure;
- automorphisms of the complex structure.

The action of the unitary group preserves all of them.

From what we have said up to now, it should be clear the  $C^*$ -algebraic formalism appears to be more general than the Schrödinger formalism which emerges via the GNS construction when we use a pure state as a starting fiducial state.

In our running example, the  $C^*$  algebra is the complex general algebra of matrices  $M_n(\mathbb{C}) \sim \mathfrak{gl}(n, \mathbb{C})$ . When we consider it as a Lie algebra and exponentiate it we get the complex general Lie group  $GL(n, \mathbb{C})$ . This group contains as a maximal compact subgroup the unitary group  $U(n)$ , whose complexification gives back the general Lie group. When the group is realized as a group of matrices, similarity transformations with respect to the linear group will take from one realization of  $U(n)$  to a different realization, and consequently, from one Hermitian structure to a different one on the vector space carrying the given realization. In finite dimensions the complex linear group and the unitary group are in one-to-one correspondence and one determines the other. Of course in infinite dimensions the situation is far from being so simple and things are not completely well defined due to the different topologies available and the lack of a properly defined differential geometry along with a proper definition of infinite dimensional Lie group. In any case, in finite dimensions the two groups do determine each other. As we have argued, for the complex projective space (the space of pure states), i.e., the two dimensional sphere for the two-level system, both groups act transitively on it. The unitary group acts by preserving the relevant structures while the complexification does not.

Within the space of mixed states the two groups have different orbits. The stratification by the rank which is natural on the space of mixed states shows that each stratum is the union of different orbits of the unitary group while each stratum is an orbit of a proper defined action of the general complex linear group which acts in a non linear manner to preserve the trace.

The infinitesimal generators of this action have an important interpretation in the framework of the Markovian dynamics for open quantum systems. Therefore we maintain that, in the realistic setting of open quantum systems, the complexification of the unitary group seems to be more relevant than the unitary group itself. With this claim in mind, it is quite natural to consider this group as the relevant group of quantum mechanics in the ideology of the Klein's programme.

In addition, if we privilege the  $C^*$ -algebra approach, the general complex linear group emerges also as the exponentiated version of the  $C^*$ -algebra thought of as a Lie algebra. In the following section we shall investigate the geometrical aspects of the general linear group.

### 3. $GL(\mathcal{H})$ AND KRAUS OPERATORS

**3.1. Geometry of  $GL(\mathcal{H})$ .** As we just saw the group  $GL(\mathcal{H})$  seems to arise naturally from the geometric structures of an open quantum system. Let us study its geometry in more detail. As in the geometrical approach we are considering the Hilbert space as a real vector space, our ingredients are a vector space  $\mathcal{H}$ , a linear space structure encoded in a Liouville operator  $\Delta$  and a complex structure  $J$  (i.e., a  $(1, 1)$  tensor field  $J$  which satisfies that  $J^2 = -\mathbb{I}$ ) compatible with  $\Delta$  in the sense that

$$(3.1) \quad \mathcal{L}_\Delta J = 0.$$

We can consider the group  $GL(\mathcal{H})$  as the (finite dimensional) subgroup of the diffeomorphism group of the set  $\mathcal{H}$  which keeps invariant the Liouville operator  $\Delta$  and the complex structure  $J$ . Thus

$$(3.2) \quad GL(\mathcal{H}) = \{\phi \in \text{Diff}(\mathcal{H}) \mid \phi_* \Delta = \Delta; \phi_* J = J\}$$

As a consequence, these transformations preserve also  $J(\Delta) = \Gamma$ . The information on the projective space is already encoded here since it arises as the foliation generated by the integrable distribution generated by  $\Delta$  and  $J(\Delta) := \Gamma$  (the global phase generator).

**3.1.1. Properties of the Lie groups.** The following facts are easy to prove

- $GL(\mathcal{H})$  contains as a maximal compact subgroup the unitary group  $U(n)$ ,
- the complexification of the group  $U(\mathcal{H})$  is isomorphic to  $GL(\mathcal{H})$ .
- In addition, introducing the tangent and cotangent bundles of the unitary group, we have

$$(3.3) \quad GL(\mathcal{H}) \rightleftarrows TU(\mathcal{H}) \rightleftarrows T^*U(\mathcal{H})$$

All three groups are symplectomorphic, the structure on  $GL(\mathcal{H})$  being defined from the product of  $U(\mathcal{H})$  and the Borel subgroup  $B(n, \mathbb{C})$  (see [1]).

We shall denote as  $\mathfrak{gl}(\mathcal{H})$  and  $\mathfrak{u}(\mathcal{H})$  the corresponding Lie algebras of these Lie groups.

**3.1.2. Properties of the Lie algebras.** In what regards the Lie algebra  $\mathfrak{gl}(\mathcal{H})$  we have also some interesting properties which will be useful later.

- As a matrix algebra,  $\mathfrak{gl}(\mathcal{H})$  is an associative algebra with involution, corresponding to the adjoint operation  $A \mapsto A^\dagger$ .
- Second,  $\mathfrak{gl}(\mathcal{H})$  carries a Hilbert space structure defined by the scalar product

$$(3.4) \quad \langle A|B \rangle = \text{Tr}(A^\dagger B)$$

If we restrict this operation to the subalgebra of the unitary group  $\mathfrak{u}(\mathcal{H})$ , it is immediate that it defines a non-degenerate metric. Therefore it can be used to define an isomorphism between the algebra and its dual

$$(3.5) \quad \hat{\cdot}: \mathfrak{u}(\mathcal{H}) \rightarrow \mathfrak{u}^*(\mathcal{H}); \quad A \mapsto \hat{A} := \langle A, \cdot \rangle \in \mathfrak{u}^*(\mathcal{H}).$$

This is the inverse mapping with respect to Eq. (2.63).

- the diffeomorphisms connecting the group  $GL(\mathcal{H})$  and the tangent and cotangent bundle of the unitary group  $U(\mathcal{H})$  have a simple translation at the level of Lie algebras:

$$(3.6) \quad \mathfrak{gl}(\mathcal{H}) \rightleftharpoons T\mathfrak{u}(\mathcal{H}) \rightleftharpoons T\mathfrak{u}^*(\mathcal{H}) \rightleftharpoons T^*\mathfrak{u}^*(\mathcal{H}) \rightleftharpoons T^*\mathfrak{u}(\mathcal{H}).$$

The new equivalences arise from the isomorphism of the unitary algebra and its dual coming from the invertibility of the metric structure defined by Eq. (3.5).

**3.2. The space of density states  $\mathcal{D}(\mathcal{H})$ .** Let us consider now the structure and properties of the space of physical states  $\mathcal{D}(\mathcal{H})$ . We know that such a space is defined as a subset of the dual of the unitary Lie algebra  $\mathfrak{u}(\mathcal{H})$ , defined as those matrices in  $\mathfrak{u}^*(\mathcal{H})$  which are positive definite and have trace equal to one, i.e.,

$$(3.7) \quad \mathcal{D}(\mathcal{H}) = \{\rho \in \mathfrak{u}^*(\mathcal{H}) \mid \rho > 0; \text{Tr} \rho = 1\}.$$

It is convenient to define first the set of positive matrices  $\mathfrak{P}(\mathcal{H})$  and impose then the constraint on the trace. Thus, we consider

$$(3.8) \quad \mathfrak{P}(\mathcal{H}) = \{\omega \in \mathfrak{u}^*(\mathcal{H}) \mid \omega = RR^\dagger \quad R \in \mathfrak{gl}(\mathcal{H})\}.$$

This view allows to derive many properties in simple terms. For instance, the left action of the group  $GL(\mathcal{H})$  is then easily written:

$$(3.9) \quad \phi : GL(\mathcal{H}) \times \mathfrak{P}(\mathcal{H}) \rightarrow \mathfrak{P}(\mathcal{H}); \quad (g, \omega) \mapsto g\omega g^\dagger.$$

On the other hand, the right action of the unitary subgroup projects onto the identity, showing that  $\mathfrak{P}(\mathcal{H})$  is the base manifold of a  $U(\mathcal{H})$ -bundle.

When considered as an action on  $\mathfrak{u}^*(\mathcal{H})$ ,  $\phi$  changes the spectrum of  $\pi$ , but it preserves its rank (signature). Indeed, it is simple to see that, along the orbit, the number of positive, negative and null eigenvalues of the elements is preserved. When considered as a bilinear form, this property corresponds to Sylvester's Law of inertia generalized to complex vector spaces (see [14] for the original proof and also [8]). If we consider the action on the set of positive operators, it is obvious that the action is inner. We can consider thus a decomposition of  $\mathfrak{P}(\mathcal{H})$  according to the rank of the state and define

$$(3.10) \quad \mathfrak{P}^k(\mathcal{H}) = \{\omega \in \mathfrak{u}^*(\mathcal{H}) \mid \omega = RR^\dagger, \quad R \in \mathfrak{gl}(\mathcal{H}); \text{rank}(\omega) = k\}.$$

Therefore  $\mathfrak{P}^k$  contains those positive elements in  $\mathfrak{u}^*(\mathcal{H})$  having  $k$  positive eigenvalues, the rest being zero. On it, we can consider also the condition on the trace and define the corresponding subset of the set  $\mathcal{D}(\mathcal{H})$ :

$$(3.11) \quad \mathcal{D}^k(\mathcal{H}) = \{\rho \in \mathfrak{P}^k(\mathcal{H}) \mid \text{Tr}(\rho) = 1\}.$$

The problem becomes more complicated when we want to consider the action of the group on the set of density operators  $\mathcal{D}(\mathcal{H})$ . It is simple to see that the action (3.9) does not preserve  $\mathcal{D}(\mathcal{H})$ . Indeed, for a general element  $g \in GL(\mathcal{H})$

$$(3.12) \quad \text{Tr}(g\rho g^\dagger) \neq \text{Tr} \rho.$$

Therefore, we must modify the action  $\phi$  to define an inner operation on the set of density states:

$$(3.13) \quad \phi^{\mathcal{D}} : GL(\mathcal{H}) \times \mathcal{D}(\mathcal{H}) \rightarrow \mathcal{D}(\mathcal{H}); \quad (g, \rho) \mapsto \frac{g\rho g^\dagger}{\text{Tr}(g\rho g^\dagger)}$$

As the action on  $\mathfrak{P}(\mathcal{H})$  preserves the rank, we know that the denominator can not vanish and therefore  $\phi^{\mathcal{D}}$  is well defined. It is immediate to verify that the action preserves the sets  $\mathcal{D}^k$ , i.e

$$(3.14) \quad \phi^{\mathcal{D}} : GL(\mathcal{H}) \times \mathcal{D}^k(\mathcal{H}) \rightarrow \mathcal{D}^k(\mathcal{H}),$$

since the action  $\phi$  preserves, by definition, the subsets of positive operators  $\mathfrak{P}^k(\mathcal{H})$ . Both manifolds  $\mathcal{D}(\mathcal{H})$  and  $\mathfrak{P}(\mathcal{H})$  become thus **stratified manifolds**, the strata being the subsets  $\mathcal{D}^k(\mathcal{H})$  and  $\mathfrak{P}^k(\mathcal{H})$ , i.e., the orbits of the corresponding action of the general linear group  $GL(n, \mathbb{C})$ .

The set  $\mathcal{D}^1(\mathcal{H})$  corresponds to those Hermitian operators which have only one eigenvalue different from zero, and therefore equal to one. These are clearly the projectors on one-dimensional subspaces of the Hilbert space  $\mathcal{H}$ , and hence they are in one-to-one correspondence with the points of the projective space  $\mathcal{P}$  by the momentum map  $\mu_{\mathcal{P}}$  introduced in Eq. (2.71). We studied in previous Section how the Hamiltonian and gradient vector fields defined different types of transformations on  $\mathcal{D}^1(\mathcal{H})$ . As we saw, the set of Hamiltonian vector fields were the fundamental vector fields corresponding to the action of the special unitary group (subgroup of  $GL(\mathcal{H}, \mathbb{C})$ ) while together with the gradient vector fields they were the fundamental vector fields of the action of the special linear group  $SL(\mathcal{H}, \mathbb{C})$ . It is not possible to consider the action of the other generators of  $GL(\mathcal{H}, \mathbb{C})$  on  $\mathcal{D}^1(\mathcal{H})$ , since the action of the generators associated to  $\hat{\sigma}_0$  (the vector fields which are associated via the momentum map  $\mu$  with vector fields  $\Delta$  and  $\Gamma$ ) act trivially on  $\mathcal{D}^1(\mathcal{H})$ .

A final comment is necessary with respect to the complex structure. On the projective space, or equivalently on the subset  $\mathcal{D}^1(\mathcal{H})$ , there is a canonical complex structure. Therefore, we know that gradient and Hamiltonian vector fields are in one-to-one correspondence defined by the tensor  $J$ . Therefore, the whole  $SL(\mathcal{H}, \mathbb{C})$ -orbit  $\mathcal{D}^1(\mathcal{H})$  can be obtained from the complexification of the generators of the unitary group (the Hamiltonian vector fields). The situation for other strata is more complicated since, although we can still obtain the stratum from the Hamiltonian and gradient vector fields, we lack of a properly defined complex structure on them to relate both sets (see [8, 9]).

**3.3. The set of density states  $\mathcal{D}(\mathcal{H})$  and Kraus operators.** As we saw in the previous section, the left action of  $\mathfrak{gl}(\mathcal{H})$  on itself corresponding to the associative product

$$(3.15) \quad \cdot : \mathfrak{gl}(\mathcal{H}) \times \mathfrak{gl}(\mathcal{H}) \rightarrow \mathfrak{gl}(\mathcal{H}); \quad (A_1, A_2) \mapsto A_1 A_2,$$

can be projected on  $\mathcal{P}(\mathcal{H})$ :

$$(3.16) \quad \cdot : \mathfrak{gl}(\mathcal{H}) \times \mathcal{P}(\mathcal{H}) \mapsto \mathcal{P}(\mathcal{H}); \quad (M, \rho) \mapsto M \rho M^\dagger$$

Given a family of operators  $\mathbb{M} = \{M_1, \dots, M_m\} \in \mathfrak{gl}(\mathcal{H}) \times \dots \times \mathfrak{gl}(\mathcal{H})$ , we can consider the action on density states as

$$(3.17) \quad (\mathbb{M}, \rho) \mapsto \sum_j M_j \rho M_j^\dagger := \mathcal{K}_{\mathbb{M}}(\rho).$$

The operator  $\mathcal{K}_{\mathbb{M}}$  shall be called a **Kraus map**. The Kraus map is said to be normalized if

$$(3.18) \quad \sum_k M_k^\dagger M_k = \mathbb{I}.$$

**Proposition 1.** *The composition of two Kraus maps is an inner operation. The set of Kraus maps endowed with the composition operation becomes a semi-group.*

$$(3.19) \quad \mathcal{K}_{\mathbb{M}} \circ \mathcal{K}_{\mathbb{M}'} = \sum_{jk} (M_j M'_k) \rho (M_j M'_k)^\dagger = \mathcal{K}_{\mathbb{M}\mathbb{M}'}.$$

By using the Hilbert space structure of  $\text{gl}(\mathcal{H})$ , we can realize a Kraus map as a sum (Jamiołkowski isomorphism [11]):

$$(3.20) \quad \mathcal{K}_{\mathbb{M}} = \sum_j |M_j\rangle\langle M_j|.$$

At this point we can claim

**Proposition 2.** *The operators of  $\text{GL}(\mathcal{H})$  form the largest subgroup of the semi-group of normalized Kraus maps.*

**Proposition 3.** *If a Kraus map  $\mathcal{K}_{\mathbb{M}}$  is invertible inside the set of Kraus maps, there exists an element  $M \in \text{GL}(\mathcal{H})$  such that*

$$(3.21) \quad \mathcal{K}_{\mathbb{M}} \rho = M \rho M^\dagger.$$

**3.4. Markovian dynamics.** In this context we can also consider the dynamics for general Markovian dynamics. It is well known that the most general markovian dynamical system takes the form of the Kossakowski-Lindblad superoperator  $L$  defined as the infinitesimal generator of a one-parameter semigroup of transformations on  $\text{u}^*(\mathcal{H})$  (see [13, 7]) which takes the form

$$(3.22) \quad L : \rho \mapsto L(\rho) := -i[H, \rho] + \frac{1}{2} \sum_{ij=1}^{N^2-1} c_{ij} ([F_i, \rho F_j^\dagger] + [F_i \rho, F_j^\dagger]),$$

where  $c_{ij}$  defines a complex positive matrix,  $H$  is Hermitian and  $F_k$  is traceless and satisfies that

$$(3.23) \quad \text{Tr}(F_i F_j^\dagger) = \delta_{ij}.$$

It is simple to see that such a transformation preserves the trace of  $\rho$  but changes its spectrum.

We can rewrite the expression of  $L$  in more geometrical terms and exhibit some of its properties in a more explicit way. Indeed, from Eq. (3.22) it is immediate that the first factor of the right hand side can be understood as a Hamiltonian vector field. The last two terms are more difficult to interpret in the form we wrote them. But let us consider an alternative factorization by re-writting them as

$$[F_i, \rho F_j^\dagger] + [F_i \rho, F_j^\dagger] = F_i \rho F_j^\dagger - \rho F_j^\dagger F_i + F_i \rho F_j^\dagger - F_j^\dagger F_i \rho = -[F_j^\dagger F_i, \rho]_+ + 2F_i \rho F_j^\dagger$$

If we consider a basis where the matrix  $c_{ij}$  is equal to the identity, and denote the corresponding eigenvectors written in terms of the operators  $\{F_k\}$  as  $V_\alpha$ , we can write the generator  $L$  in the form

$$(3.24) \quad L(\rho) = -i[H, \rho] - \frac{1}{2} \sum_{\alpha=1}^{N^2-1} ([V_\alpha^\dagger V_\alpha, \rho] - 2V_\alpha \rho V_\alpha^\dagger) = -i[H, \rho] - \frac{1}{2} [G, \rho]_+ + \sum_{\alpha=1}^{N^2-1} V_\alpha \rho V_\alpha^\dagger,$$

where

$$(3.25) \quad G = \sum_{\alpha} V_\alpha^\dagger V_\alpha.$$

We easily recognize in Eq. (3.24) the three types of transformations we have presented in this paper:

- the first term defines a Hamiltonian vector field,
- the second corresponds to a gradient vector field
- and finally, the third represents the action of the Kraus map  $\mathbb{V} = \{V_1, \dots, V_{N^2-1}\}$ .

The Hamiltonian vector field alone generates a unitary transformation. The second and third terms are the ones responsible for the breaking of unitarity in the markovian evolution. But they break unitarity in such a way that the total transformation generated by  $L$  does not change the trace of  $\rho$ .

Summarizing we can say that the generators of the actions of the group  $GL(\mathcal{H})$  on  $\mathcal{D}(\mathcal{H}) \subset \mathfrak{u}^*(\mathcal{H})$  allow us to express in geometrical terms the most general form of markovian evolution of a finite dimensional quantum system.

#### 4. CONCLUSIONS AND OUTLOOK

We have seen that the markovian dynamical evolution of an open system is not associated with a group of transformations but with a semi-group. The maximal subgroup is the general linear group which we interpret as the relevant group of Quantum Mechanics and the one identified according to the Klein programme when dealing with open systems. A natural question arises: is it possible to generalize Klein's programme to semi-groups?

Another interesting possibility arises from the fact that the  $C^*$ -algebra which plays a crucial role in our presentation is also a groupoid algebra (see references [10]). Is it possible to consider an extension of the Erlangen Programme to groupoids?

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BIFI-DEPARTAMENTO DE FÍSICA TEÓRICA, EDIFICIO I+D-CAMPUS RÍO EBRO, UNIVERSIDAD DE ZARAGOZA, 50018 ZARAGOZA(SPAIN),  
*E-mail address:* `jesus.clementegallardo@bifi.es`

DIPARTIMENTO DE FISICA, UNIVERSITÀ FEDERICO II DI NAPOLI AND, INFN SEZIONE DI NAPOLI, 80126-NAPOLI (ITALY)  
*E-mail address:* `marmo@na.infn.it`